INTEGRATION OF FUNCTIONS OF Several VARIABLES

We begin with the Double Integral for a function of two variables.

RECALL how the Definite Integral $\int_a^b f(x)\,dx$ works.

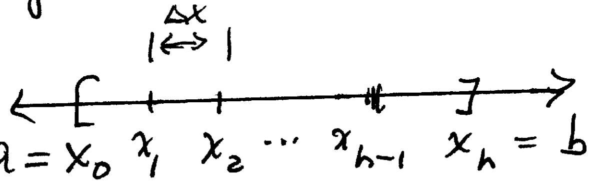
$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \left(\sum_{i=1}^{n} f(x_i^*)\,\Delta x \right)$$

a Riemann Sum

and $\Delta x = \frac{b-a}{n}$

Each Riemann Sum $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$

is calculated by dividing the interval $[a,b]$ into n subintervals



For each i, $1 \le i \le n$, the width of $[x_{i-1}, x_i]$ is $\Delta x = (b-a)/n$

For each i, $1 \le i \le n$, select x_i^* in $[x_{i-1}, x_i]$.

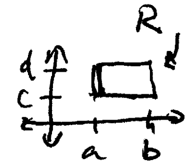
The n^{th} Riemann Sum is $\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$.

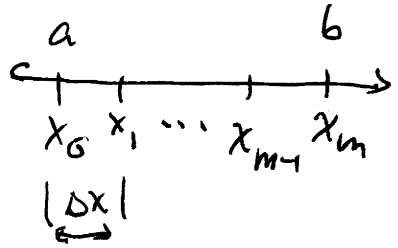
When the function $f(x) \ge 0$ for all x in $[a,b]$,

$A = \int_a^b f(x)\,dx =$ the Area under the curve.

Now, Double Integrals

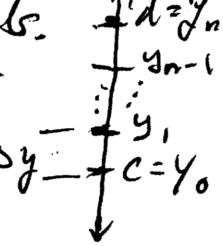
In the xy plane, Let R = The Rectangle $[a, b] \times [c, d]$





Subdivide $[a, b]$ into m subintervals.

Subdivide $[c, d]$ into n Subintervals



For each i, $1 \le i \le m$, and for each j, $1 \le j \le n$,

R_{ij} = the subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$,

and its <u>area</u> $\Delta A_{ij} = (\Delta x)(\Delta y)$.

Let $z = f(x, y)$ be a function of 2 variables.

For each i, $1 \le i \le m$, and for each j, $1 \le j \le n$,

select point (x^*_{ij}, y^*_{ij}) in R_{ij} and

Compute $\displaystyle\sum_{i=1}^{m} \left(\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \cdot \Delta A_{ij} \right)$,

area of R_{ij}

a # called the Double Rieman Sum over R.

EXAMPLE: Let $f(x,y) = x + 2y + 1$.

Let $R = [0,2] \times [1,3]$. Let $m = 4$ and $n = 2$

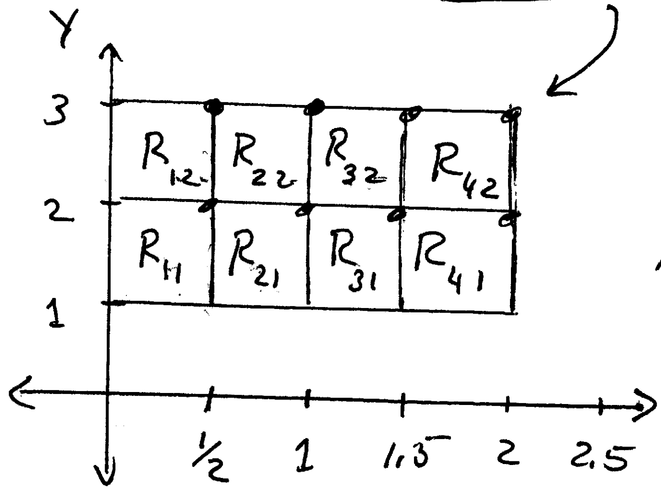
So $\Delta x = \frac{2-0}{4} = \frac{1}{2}$

$\Delta y = \frac{3-1}{2} = 1$

$\Delta A_{11} = \frac{1}{2} \cdot 1 = \frac{1}{2}$

$\Delta A_{22} = \frac{1}{2} \cdot 1 = \frac{1}{2}$

... etc



SELECTING (x^*_{ij}, y^*_{ij})

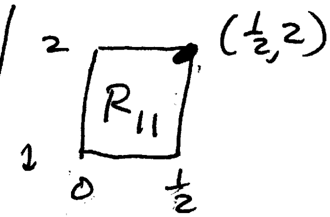
Method 1: The Upper-Right-hand-corner method.

$(x^*_{11}, y^*_{11}) = (\frac{1}{2}, 2)$

$(x^*_{12}, y^*_{12}) = (\frac{1}{2}, 3)$

$\vdots \qquad \vdots$

$(x^*_{42}, y^*_{42}) = (2, 3)$



Recall $f(x,y) = x + 2y + 1$

The "4x2" Riemann Sum is

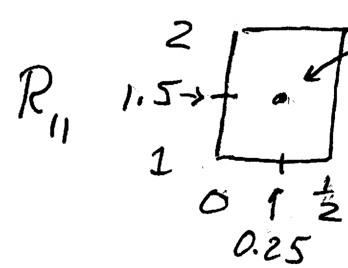
$$\sum_{i=1}^{4} \left(\sum_{j=1}^{2} \left(f(x^*_{ij}, y^*_{ij}) \cdot \Delta A_{ij} \right) \right)$$

$$= (\tfrac{1}{2}+4+1)(\tfrac{1}{2}) + (\tfrac{1}{2}+6+1)(\tfrac{1}{2}) + \cdots + (2+6+1)(\tfrac{1}{2})$$

$$= 29$$

Method 2: The Midpoint Rule:

(x^*_{ij}, y^*_{ij}) = The middle point in R_{ij}



$(x^*_{11}, y^*_{11}) = (.25, 1.5)$

Using the Midpoint Rule here, The "4x2" Riemann Sum is

$$\sum_{i=1}^{4} \left(\sum_{j=1}^{2} \left(f(x^*_{ij}, y^*_{ij}) \cdot \Delta A_{ij} \right) \right)$$

$$= 24$$

Now, the Double Integral of f

over Rectangle R is the Number which is

$$\iint_R f(x,y)\,dA = \lim_{\substack{m\to\infty \\ n\to\infty}} \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A_i \right)\right)$$

In this example, where $f(x) = x + 2y + 1$ and $R = [0,2] \times [1,3]$, the Double Integral exists

and $\iint_R (x+2y+1)\,dA = 24$

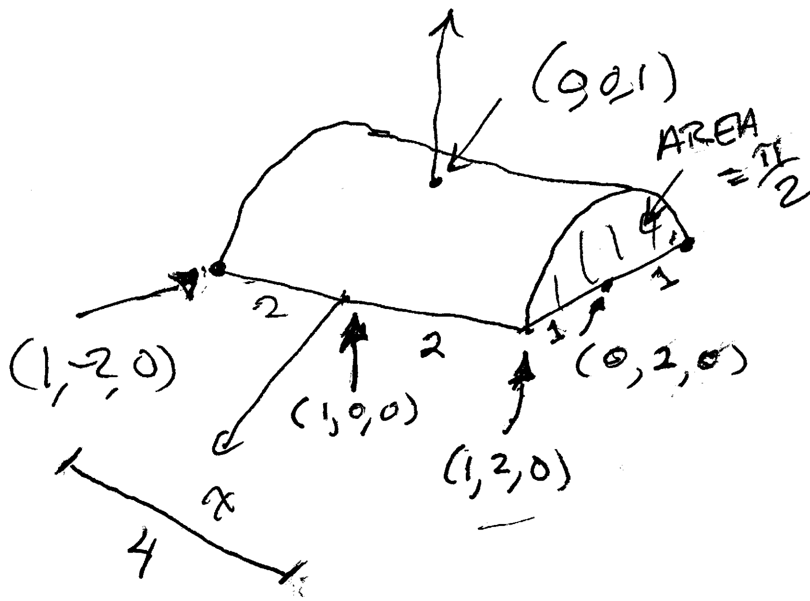
[The fact that the midpoint rule of selecting the (x_{ij}^*, y_{ij}^*) points gave a Riemann Sum that actually equals the Double Integral Value is a result of the fact that the graph of the function is a plane, a tilted plane, and the top of each "Riemann Sum Column" with volume ΔV_{ij} is horizontally flat, and the top of the solid above the subrectangle is flat at an angle — so the column rises above the angled "roof" just as much as it falls below the the "roof," so the sum of the volumes of the columns equals the volume of the solid.].

If $R = \{(x,y) \mid -1 \le x \le 1,\ -2 \le y \le 2\}$,

$[-1,-1] \times [-2,2]$

evaluate the Double Integral

$$\iint_R \sqrt{1-x^2}\, dA$$



$z = \sqrt{1-x^2} = f(x,y)$

$\iint_R \sqrt{1-x^2}\, dA$ = Volume under this HALF-Cylinder

Volume $= \frac{\pi}{2} \times 4 = 2\pi$